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Analytical solution for a reinforcement layer bonded to an elliptic hole under a remote uniform load

C.K. Chao ^{a,*}, L.M. Lu ^a, C.K. Chen ^a, F.M. Chen ^b^a Department of Mechanical Engineering, National Taiwan University of Science and Technology, 43 Keelung Road, Section 4, Taipei 106, Taiwan, ROC^b Department of Mechanical Engineering, Nan-Kai University of Technology, 568 Chung Cheng Road, Tsao Tun 542, Taiwan, ROC

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ABSTRACT

A general solution to a reinforced elliptic hole embedded in an infinite matrix subjected to a remote uniform load is provided in this paper. Investigations on the present elasticity problem are rather tedious due to the presence of material inhomogeneities and complex geometric configurations. Based on the technique of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the general expressions of the displacement and stresses in a reinforcement layer and the matrix are derived explicitly in a series form. Some numerical results are provided to investigate the effects of the material combinations and geometric configurations on the interfacial stresses. The results show that there exists an optimum design of a reinforcement layer such that both the magnitude of stress concentration and the interfacial stresses could be fairly reduced.

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1. Introduction

The determination of stresses for the elliptic notch problem has received extensive attention from investigators in the field of plane elasticity. Both experimental and analytical studies on this subject can be found in the literature. Early work by Muskhelishvili (1953) gave the details of the stress field around the elliptical cavity under uniform loading for an isotropic and homogeneous material. Uniaxial compressive experiments were performed by Cotterell (1972) on plates of annealed glass with an elliptic notch. Experiments in uniaxial tension were conducted by Erdogan and Sih (1963), Williams and Ewing (1972) on PMMA, and by Pook (1971) on aluminum plates with a slit crack; and by Wu et al. (1977), on PMMA plates with an angled elliptical notch. As to the thermoelastic problem, Florence and Goodier (1960) provided an exact solution for an isotropic medium containing a circular or oval hole by the method of England (1971). Chen (1967) solved the thermal stresses for orthotropic medium with an elliptic hole based on the complex variable technique developed by Green and Zerna (1954). Tarn and Wang (1993) gave a closed form solution for anisotropic materials with a hole or a rigid inclusion, based on the Lekhnitskii complex potential approach (1963). Using Stroh formalism (1958), Hwu (1990) found the thermal stresses for anisotropic body with an elliptic hole. Based on the Lekhnitskii complex potential approach, Chao and Shen (1998) provided an exact solution of thermal stresses for an anisotropic body with an elliptic inclusion.

Stress concentration at an elliptic hole induced by a remote uniform load is one of the classical problems in linear elasticity. To reduce stress concentration, an effective method is applying a reinforcement layer, with appropriate geometry and material property, bonded to a hole (Chen and Chao, 2008). This method has been widely applied to many practical problems where the elastic mismatch-induced stresses are of vital importance to mechanical integrity. In doing so, the problem is reduced to one of stresses within a two-phase elliptical composite. Unfortunately, the stress field within a two-phase elliptical composite is quite complicated. That is why the existing related works are mainly limited to a multiple-phase circular composite (Chao et al., 2006). When a reinforcement layer is introduced in the analysis, the interfacial stresses between a reinforcement layer and the matrix must be taken into account. In this work, we consider a reinforced elliptical hole in an infinite plate subject to a remote uniform load. The reinforcement layer is bounded by two confocal ellipses. The proposed method is based on the method of conformal mapping and the technique of analytical continuation that is alternately applied across two concentric circles. The plan of this paper is as follows. The general formulation for plane elasticity and the method of conformal mapping are provided in Section 2. The series form solutions of the complex potentials of the stresses are given in Section 3. Some numerical examples are solved in Section 4. Finally, Section 5 concludes the article.

2. Problem formulation

Consider a reinforced elliptical hole in an unbounded matrix subjected to a remote uniform load (see Fig. 1). Let Ω_1 denote

* Corresponding author. Fax: +886 2 2737 6460.

E-mail address: ckchao@mail.ntust.edu.tw (C.K. Chao).

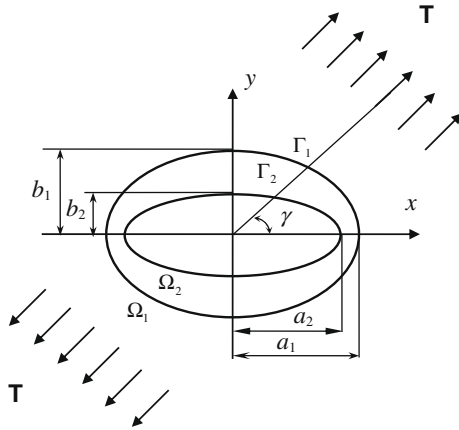


Fig. 1. A reinforced elliptic hole in an infinite plate subjected to a remote uniform load.

the matrix, and let Ω_2 denote the reinforcement layer. The boundaries of the reinforcement are two confocal ellipses Γ_1, Γ_2 with a_1, a_2 and b_1, b_2 being the semimajor and semiminor axes for each, respectively.

It is well known that for plane deformations, the displacement components (u_x, u_y), stress components ($\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$) and the components of the resultant force (X, Y) are given in terms of two holomorphic functions $\phi(z)$ and $\psi(z)$ (Muskhelishvili, 1953);

$$2G(u_x + iu_y) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \quad (1)$$

$$\sigma_{xx} + \sigma_{yy} = 2[\phi'(z) + \overline{\phi'(z)}] \quad (2)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \quad (3)$$

$$-Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \quad (4)$$

where G is the shear modulus, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress with ν being the Poisson's ratio. Here a superimposed bar represents the complex conjugate.

The boundary stresses are written in normal–tangential ((n, t) –) coordinates as:

$$\sigma_{nn} + i\sigma_{nt} = \phi'(z) + \overline{\phi'(z)} - [z\overline{\phi''(z)} + \overline{\psi'(z)}]e^{-2i\alpha(z)} \quad (5)$$

where n is the outward unit normal at the boundary which is also represented, in complex form, by $e^{i\alpha(z)}$ (where α defines the angle between the normal direction n and the positive x -axis).

Now we introduce the following mapping function

$$z = m(\zeta) = \frac{l}{2} \left[R\zeta + \frac{1}{R\zeta} \right], \quad R\zeta = \frac{z}{l} \left\{ 1 + \left[1 - \left(\frac{l}{z} \right)^2 \right]^{1/2} \right\}, \quad (6)$$

$$\zeta = \xi + i\eta = re^{i\theta}$$

where $R = \sqrt{\frac{a_2+b_2}{a_2-b_2}}$ and $l = \sqrt{a_2^2 - b_2^2}$.

This mapping function maps the confocal ellipses Γ_1, Γ_2 in the z -plane onto the concentric circles L_1, L_2 in the ζ -plane with radii ρ_1, ρ_2 (see Fig. 2)

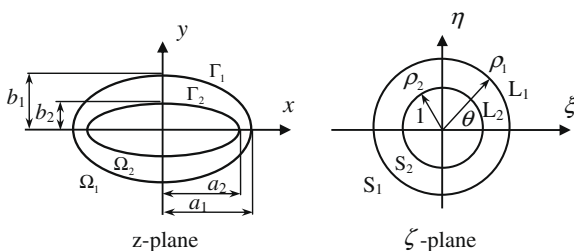


Fig. 2. Conformal mapping from the z -plane to the ζ -plane.

$$\rho_i = \frac{a_i + b_i}{a_2 + b_2} \quad i = 1, 2$$

For convenience of calculation, we write $\phi(\zeta) = \phi(m(\zeta))$ and $\psi(\zeta) = \psi(m(\zeta))$ so that in the mapped ζ -plane, the displacements, stresses and resultant forces take the form

$$2G(u_x + iu_y) = \kappa\phi(\zeta) - \frac{m(\zeta)}{m'(\zeta)}\overline{\phi'(\zeta)} - \overline{\psi(\zeta)} \quad (7)$$

$$\sigma_{xx} + \sigma_{yy} = 2 \left\{ \frac{\phi'(\zeta)}{m'(\zeta)} + \frac{\overline{\phi'(\zeta)}}{\overline{m'(\zeta)}} \right\} \quad (8)$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2 \left[\frac{m(\zeta)}{m'(\zeta)} \frac{d}{d\zeta} \left\{ \frac{\phi'(\zeta)}{m'(\zeta)} \right\} + \frac{\psi'(\zeta)}{m'(\zeta)} \right]. \quad (9)$$

$$-Y + iX = \phi(\zeta) + \frac{m(\zeta)}{m'(\zeta)}\overline{\phi'(\zeta)} + \overline{\psi(\zeta)} \quad (10)$$

In the mapped plane, noting the following relation from (England, 1971);

$$e^{2i\alpha(z)} = \frac{\zeta}{\bar{\zeta}} \frac{m(\zeta)}{m'(\zeta)}$$

Eq. (5) becomes

$$\sigma_{nn} + i\sigma_{nt} = \left\{ \frac{\phi'(\zeta)}{m'(\zeta)} + \frac{\overline{\phi'(\zeta)}}{\overline{m'(\zeta)}} \right\} - \left[\frac{m(\zeta)}{m'(\zeta)} \frac{d}{d\zeta} \left\{ \frac{\phi'(\zeta)}{m'(\zeta)} \right\} + \frac{\psi'(\zeta)}{m'(\zeta)} \right] \frac{\zeta m'(\zeta)}{\bar{\zeta} m'(\zeta)} \quad (11)$$

3. Stress field

In this section, we will derive the stress fields for a reinforced elliptic hole in an infinite plate subjected to a remote uniform tension. The solution for a homogeneous infinite plate subjected to a remote uniform tension T acting with an angle γ to the x -axis can be trivially given as

$$\phi_0(\zeta) = \frac{T}{4}\zeta \quad (12)$$

$$\psi_0(\zeta) = -\frac{Te^{-2i\gamma}\zeta}{2} \quad (13)$$

For a region bounded by a circle, say $c = |\zeta|$, we introduce an auxiliary stress function $\omega(\zeta)$ such that

$$\omega(\zeta) = \frac{\bar{m}(\frac{c^2}{\zeta})}{m'(\zeta)}\phi'(\zeta) + \psi(\zeta) \quad (14)$$

Unlike the standard Muskhelishvili complex functions $\phi(\zeta)$ and $\psi(\zeta)$, the function is dependent on the radius of any circular interface. Substitution of Eq. (14) into Eqs. (7) and (10), respectively, yields

$$2G(u_x + iu_y) = \kappa\phi(\zeta) - \overline{\omega(\zeta)} \quad |\zeta| = c$$

and

$$-Y + iX = \phi(\zeta) + \overline{\omega(\zeta)} \quad |\zeta| = c$$

In view of Eqs. (6), (12), (13) and (14), we have

$$\omega_0(\zeta) = \omega_{0a}(\zeta) + \omega_{0b}(\zeta)$$

where $\omega_{0a}(\zeta) = \frac{\zeta}{R^2\rho_1^2} \frac{T}{4} - \frac{T}{2}e^{-2i\gamma}\zeta$ and $\omega_{0b}(\zeta) = \frac{R^2\rho_1^2}{R^2\rho_1^2} \frac{\zeta}{4} - \frac{\zeta}{R^2\rho_1^2} \frac{T}{4}$ which are, respectively, holomorphic in $|\zeta| \leq \rho_1$ and $|\zeta| \geq \rho_1$.

The stress functions can be assumed as

$$\phi(\zeta) = \begin{cases} \phi_0(\zeta) + \sum_{n=1}^{\infty} \phi_n(\zeta) & \zeta \in S_1 \\ \sum_{n=1}^{\infty} \phi_{an}(\zeta) + \sum_{n=1}^{\infty} \phi_{bn}(\zeta) & \zeta \in S_2 \end{cases} \quad (15)$$

$$\omega(\zeta) = \begin{cases} \omega_0(\zeta) + \sum_{n=1}^{\infty} \omega_n(\zeta) & \zeta \in S_1 \\ \sum_{n=1}^{\infty} \omega_{an}(\zeta) + \sum_{n=1}^{\infty} \omega_{bn}(\zeta) & \zeta \in S_2 \end{cases} \quad (16)$$

The alternating technique and the analytical continuation method are applied to derive the unknown stress functions as follows.

Step 1: Analytical continuation across L_1

Two pairs of stress functions $\phi_{a1}(\zeta), \omega_{a1}(\zeta)$ holomorphic in $|\zeta| \leq \rho_1$ and $\phi_1(\zeta), \omega_1(\zeta)$ holomorphic in $|\zeta| \geq \rho_1$ are introduced to satisfy the continuity conditions along L_1 , such that

$$\phi_1(\sigma) + \overline{\omega_1(\sigma)} + \phi_0(\sigma) + \overline{\omega_0(\sigma)} = \phi_{a1}(\sigma) + \overline{\omega_{a1}(\sigma)} \quad \sigma \in L_1 \quad (17)$$

$$\begin{aligned} \frac{\kappa_1}{G_1} [\phi_1(\sigma) + \phi_0(\sigma)] - \frac{1}{G_1} [\overline{\omega_1(\sigma)} + \overline{\omega_0(\sigma)}] \\ = \frac{\kappa_2}{G_2} \phi_{a1}(\sigma) - \frac{1}{G_2} \overline{\omega_{a1}(\sigma)} \quad \sigma \in L_1 \end{aligned} \quad (18)$$

By the standard analytical continuation arguments, it follows that

$$\phi_1(\zeta) + \overline{\omega_{0a}}\left(\frac{\rho_1^2}{\zeta}\right) - \overline{\omega_{a1}}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} c_1 = 0 \quad \zeta \in S_1 \quad (19)$$

$$\phi_{a1}(\zeta) - \phi_0(\zeta) - \overline{\omega_{0b}}\left(\frac{\rho_1^2}{\zeta}\right) - \overline{\omega_1}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} c_1 = 0 \quad \zeta \in S_2 \quad (20)$$

$$\frac{\kappa_1}{G_1} \phi_1(\zeta) - \frac{1}{G_1} \overline{\omega_{0a}}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{1}{G_2} \overline{\omega_{a1}}\left(\frac{\rho_1^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} \frac{c_1}{G_2} = 0 \quad \zeta \in S_1 \quad (21)$$

$$\begin{aligned} \frac{\kappa_2}{G_2} \phi_{a1}(\zeta) - \frac{\kappa_1}{G_1} \phi_0(\zeta) + \frac{1}{G_1} \overline{\omega_{0b}}\left(\frac{\rho_1^2}{\zeta}\right) \\ + \frac{1}{G_1} \overline{\omega_1}\left(\frac{\rho_1^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} \frac{c_1}{G_2} = 0 \quad \zeta \in S_2 \end{aligned} \quad (22)$$

where $c_1 = \overline{\phi'_{a1}}\left(\frac{1}{R}\right)$.

Solve Eqs. (19)–(22) to yield

$$\phi_1(\zeta) = \Pi_{21} \overline{\omega_{0a}}\left(\frac{\rho_1^2}{\zeta}\right) \quad \zeta \in S_1 \quad (23)$$

$$\phi_{a1}(\zeta) = (1 + \Lambda_{21}) \phi_0(\zeta) + \Pi_{12} \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} c_1 \quad \zeta \in S_2 \quad (24)$$

$$\omega_{a1}(\zeta) = (1 + \Pi_{21}) \omega_{0a}(\zeta) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\frac{\rho_1^2}{\zeta}}{1 - \frac{1}{R^2\zeta^2}} \overline{c_1} \quad \zeta \in S_2 \quad (25)$$

$$\omega_1(\zeta) = \Lambda_{21} \overline{\phi_0}\left(\frac{\rho_1^2}{\zeta}\right) - \omega_{0b}(\zeta) + (1 + \Pi_{12}) \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\frac{\rho_1^2}{\zeta}}{1 - \frac{1}{R^2\zeta^2}} \overline{c_1} \quad \zeta \in S_1 \quad (26)$$

where $\Pi_{12} = \frac{G_1 - G_2}{\kappa_2 G_1 + G_2}$ and $\Lambda_{12} = \frac{G_1 \kappa_2 - G_2 \kappa_1}{G_1 + G_2 \kappa_1}$.

Step 2: Analytical continuation across L_2

Since $\phi_{a1}(\zeta)$ and $\omega_{a1}(\zeta)$ cannot satisfy the traction-free boundary condition at L_2 , two functions, $\phi_{b1}(\zeta)$ and $\omega_{b1}(\zeta)$, holomorphic in $|\zeta| \geq \rho_2$, are introduced, such that

$$\phi_{a1}(\sigma) + \overline{\omega'_{a1}(\sigma)} + \phi_{b1}(\sigma) + \overline{\omega_{b1}(\sigma)} = 0 \quad (27)$$

where

$$\omega'_{a1}(\zeta) = \frac{\frac{R}{\zeta}(\rho_2^2 - \rho_1^2) + \frac{\zeta}{R}\left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2}\right)}{R - \frac{1}{R\zeta^2}} \phi'_{a1}(\zeta) + \omega_{a1}(\zeta)$$

By the analytical continuation method, we have

$$\phi_{a1}(\zeta) + \frac{\left(\frac{1}{R^4\rho_2^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_2^4}} c_1 + \overline{\omega_{b1}}\left(\frac{\rho_2^2}{\zeta}\right) = 0 \quad (28)$$

$$\overline{\omega'_{a1}}\left(\frac{\rho_2^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_2^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_2^4}} c_1 + \phi_{b1}(\zeta) = 0 \quad (29)$$

$$\phi_{b1}(\zeta) = -\overline{\omega'_{a1}}\left(\frac{\rho_2^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_2^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_2^4}} c_1 \quad \zeta \in S_2 \quad (30)$$

$$\omega_{b1}(\zeta) = -\overline{\phi_{a1}}\left(\frac{\rho_2^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_2^4} + 1\right)\frac{\rho_2^2}{\zeta}}{1 - \frac{\rho_2^4}{R^2\zeta^2}} \overline{c_1} \quad \zeta \in S_2 \quad (31)$$

Step 3: Analytical continuation across L_1

Since $\phi_{b1}(\zeta)$ and $\omega_{b1}(\zeta)$ cannot satisfy the continuity condition at L_2 , two pairs of stress functions $\phi_{a2}(\zeta), \omega_{a2}(\zeta)$ and $\phi_2(\zeta), \omega_2(\zeta)$, respectively, holomorphic in $|\zeta| \leq \rho_1$ and $|\zeta| \geq \rho_1$ are introduced to satisfy the continuity conditions along L_1 , such that

$$\phi_2(\sigma) + \overline{\omega_2(\sigma)} = \phi_{b1}(\sigma) + \overline{\omega'_{b1}(\sigma)} + \phi_{a2}(\sigma) + \overline{\omega_{a2}(\sigma)} \quad \sigma \in L_1 \quad (32)$$

$$\begin{aligned} \frac{\kappa_1}{G_1} \phi_2(\sigma) - \frac{1}{G_1} \overline{\omega_2(\sigma)} = \frac{\kappa_2}{G_2} \phi_{b1}(\sigma) - \frac{1}{G_2} \overline{\omega'_{b1}(\sigma)} \\ + \frac{\kappa_2}{G_2} \phi_{a2}(\sigma) - \frac{1}{G_2} \overline{\omega_{a2}(\sigma)} \quad \sigma \in L_1 \end{aligned} \quad (33)$$

where

$$\omega'_{b1}(\zeta) = \omega_{b1}(\zeta) + \frac{\frac{R(\rho_1^2 - \rho_2^2)}{\zeta} + \frac{\zeta}{R}\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2}\right)}{R - \frac{1}{R\zeta^2}} \phi'_{b1}(\zeta)$$

By the standard analytical continuation arguments, it follows that

$$\phi_2(\zeta) - \phi_{b1}(\zeta) - \overline{\omega_{a2}}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} c_2 = 0 \quad \zeta \in S_1 \quad (34)$$

$$\phi_{a2}(\zeta) + \overline{\omega'_{b1}}\left(\frac{\rho_1^2}{\zeta}\right) - \overline{\omega_2}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} c_2 = 0 \quad \zeta \in S_2 \quad (35)$$

$$\frac{\kappa_1}{G_1} \phi_2(\zeta) - \frac{\kappa_2}{G_2} \phi_{b1}(\zeta) + \frac{1}{G_2} \overline{\omega_{a2}}\left(\frac{\rho_1^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} \frac{c_2}{G_2} = 0 \quad \zeta \in S_1 \quad (36)$$

$$\frac{\kappa_2}{G_2} \phi_{a2}(\zeta) - \frac{1}{G_2} \overline{\omega'_{b1}}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{1}{G_1} \overline{\omega_2}\left(\frac{\rho_1^2}{\zeta}\right) - \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}} \frac{c_2}{G_2} = 0 \quad \zeta \in S_2 \quad (37)$$

where $c_2 = \overline{\phi'_{a2}}\left(\frac{1}{R}\right)$.

Solve Eqs. (34)–(37) to yield

$$\phi_2(\zeta) = (1 + \Lambda_{12})\phi_{b1}(\zeta) \quad \zeta \in S_1 \quad (38)$$

$$\omega_2(\zeta) = (1 + \Pi_{12})\omega_{b1}^I(\zeta) + (1 + \Pi_{12})\frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\frac{\rho_1^2}{\zeta}}{1 - \frac{1}{R^2\zeta^2}}\bar{c}_2 \quad \zeta \in S_1 \quad (39)$$

$$\phi_{a2}(\zeta) = \Pi_{12}\bar{\omega}_{b1}^I\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}}\Pi_{12}c_2 \quad \zeta \in S_2 \quad (40)$$

$$\omega_{a2}(\zeta) = \Lambda_{12}\bar{\phi}_{b1}\left(\frac{\rho_1^2}{\zeta}\right) + \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\frac{\rho_1^2}{\zeta}}{1 - \frac{1}{R^2\zeta^2}}\bar{c}_2 \quad \zeta \in S_2 \quad (41)$$

Repetitions of the previous two steps are made until arriving at the results which have satisfied the continuity condition and the boundary condition. Finally, one can express all the functions $\phi_n(\zeta)$, $\omega_n(\zeta)$, $\phi_{an}(\zeta)$, $\omega_{an}(\zeta)$, $\phi_{bn}(\zeta)$ and $\omega_{bn}(\zeta)$ ($n=2,3,4,\dots$) in terms of $\phi_{a1}(\zeta)$ and $\phi_{b1}(\zeta)$ as follows

$$\left\{ \begin{array}{l} \phi_{an}(\zeta) = -\Pi_{12}\phi_{a(n-1)}\left[\left(\frac{\rho_2^2}{\rho_1^2}\zeta\right)\right] \\ \quad + \Pi_{12}\bar{t}_{21}\left(\frac{\rho_1^2}{\zeta}\right)\phi'_{b(n-1)}\left(\frac{\rho_1^2}{\zeta}\right) \\ \quad + \Pi_{12}\left[S^I(\zeta)c_n - S^{II}\left(\frac{\rho_2^2}{\rho_1^2}\zeta\right)c_{n-1}\right] \\ \omega_{an}(\zeta) = \Lambda_{12}\bar{\phi}_{b(n-1)}\left(\frac{\rho_1^2}{\zeta}\right) + \bar{S}^I\left(\frac{\rho_1^2}{\zeta}\right)\bar{c}_n \\ \phi_n(\zeta) = (1 + \Lambda_{12})\phi_{b(n-1)}(\zeta) \\ \omega_n(\zeta) = -(1 + \Pi_{12})\bar{\phi}_{a(n-1)}\left(\frac{\rho_2^2}{\zeta}\right) \\ \quad + (1 + \Pi_{12})\bar{t}_{21}(\zeta)\phi'_{b(n-1)}(\zeta) \\ \quad + (1 + \Pi_{12})\left[\bar{S}^I\left(\frac{\rho_1^2}{\zeta}\right)\bar{c}_n - \bar{S}^{II}\left(\frac{\rho_2^2}{\zeta}\right)\bar{c}_{n-1}\right] \end{array} \right. \quad \text{for } n = 2, 3, 4, \dots \quad (42)$$

$$\left\{ \begin{array}{l} \phi_{bn}(\zeta) = -\Lambda_{12}\phi_{b(n-1)}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right] \\ \quad + \Pi_{12}\left(\frac{\rho_2}{\rho_1}\right)^2\bar{t}_{12}\left(\frac{\rho_2^2}{\zeta}\right)\phi'_{a(n-1)}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\frac{\rho_2^2}{\zeta}\right] \\ \quad + \Pi_{12}\frac{\rho_1^2\zeta^2}{\rho_2^2}\bar{t}_{12}\left(\frac{\rho_2^2}{\zeta}\right)t'_{21}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right]\phi'_{b(n-1)}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right] \\ \quad + \Pi_{12}\frac{\rho_1^2\zeta^2}{\rho_2^2}\bar{t}_{12}\left(\frac{\rho_2^2}{\zeta}\right)t_{21}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right]\phi''_{b(n-1)}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right] \\ \quad + c_n\left[S^{II}(\zeta) - S^I\left(\frac{\rho_2^2}{\rho_1^2}\zeta\right)\right] \\ \quad + \Pi_{12}\bar{c}_{n-1}\frac{\rho_2^2}{\rho_1^2}\bar{t}_{12}\left(\frac{\rho_2^2}{\zeta}\right)\bar{S}^{III}\left(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta}\right) \\ \quad - \Pi_{12}\bar{c}_n\bar{t}_{12}\left(\frac{\rho_2^2}{\zeta}\right)\bar{S}^{IV}\left(\frac{\rho_2^2}{\zeta}\right) \\ \omega_{bn}(\zeta) = \Pi_{12}\bar{\phi}_{a(n-1)}\left[\left(\frac{\rho_2}{\rho_1}\right)^2\frac{\rho_2^2}{\zeta}\right] \\ \quad - \Pi_{12}t_{21}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right]\phi'_{b(n-1)}\left[\left(\frac{\rho_1}{\rho_2}\right)^2\zeta\right] - \bar{c}_n\bar{S}^{II}\left(\frac{\rho_2^2}{\zeta}\right) \\ \quad - \Pi_{12}\left[\bar{S}^I\left(\frac{\rho_2^2}{\zeta}\right)\bar{c}_n - \bar{S}^{II}\left(\frac{\rho_2^2}{\rho_1^2}\frac{\rho_2^2}{\zeta}\right)\bar{c}_{n-1}\right] \end{array} \right. \quad \text{for } n = 2, 3, 4, \dots \quad (43)$$

where

$$S^I(\zeta) = \frac{\left(\frac{1}{R^4\rho_1^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_1^4}}$$

$$S^{II}(\zeta) = \frac{\left(\frac{1}{R^4\rho_2^4} + 1\right)\zeta}{1 - \frac{\zeta^2}{R^2\rho_2^4}}$$

$$t_{ij}(\zeta) = \frac{\frac{R(\rho_j^2 - \rho_i^2)}{\zeta} + \frac{\zeta}{R}\left(\frac{1}{\rho_j^2} - \frac{1}{\rho_i^2}\right)}{R - \frac{1}{R\zeta^2}} \quad i = 1, 2; \quad j = 1, 2$$

$$c_n = \bar{\phi}'_{an}\left(\frac{1}{R}\right)$$

For the limiting case when the regions S_1 and S_2 are made of the same material for the corresponding elliptical hole problem, Eqs. (15) and (16) can be simplified to an exact form given by (see Appendix A)

$$\phi(\zeta) = \frac{T}{4}\zeta + \frac{T}{4}\left[\frac{2e^{2i\gamma}}{\zeta} - \frac{m}{\zeta}\right] \quad (44)$$

$$\psi(\zeta) = -\frac{T}{2}e^{-2i\gamma}\zeta - \frac{T}{4}\frac{1}{\zeta} - \frac{(m^2 + 1)\zeta}{\zeta^2 - m}\frac{T}{4} - \frac{\zeta + m\zeta^3}{\zeta^2 - m}\left(\frac{m}{\zeta^2}\frac{T}{4} - \frac{T}{2}e^{2i\gamma}\frac{1}{\zeta^2}\right) \quad (45)$$

where $m = \frac{a_2 - b_2}{a_2 + b_2}$.

The above solutions given in (44) and (45) are the same as the results provided by England (1971).

4. Results and discussion

All the stress functions $\phi_n(\zeta)$, $\omega_n(\zeta)$, $\phi_{an}(\zeta)$, $\omega_{an}(\zeta)$, $\phi_{bn}(\zeta)$ and $\omega_{bn}(\zeta)$ in Eqs. (42) and (43) can be calculated from $\phi_{a1}(\zeta)$ and $\phi_{b1}(\zeta)$. The rate of the convergence depends on the two non-dimensional bimaterial constants Λ_{12} and Π_{12} . For most combinations of materials, Λ_{12} and Π_{12} are less than 1 and 0.5, respectively, which guarantees rapid convergence. The angular variations of the interfacial normal stress and interfacial shear stress between the reinforcement layer and the matrix, under the condition that a uniform tensile load is applied along the y axis, are shown in Figs. 3 and 4, respectively. As expected, the tangential normal stress is symmetric about the y-axis while the tangential shear stress is asymmetric about the y-axis. The magnitudes of both the normal stress and the shear stress increase with an increasing ratio of G_2/G_1 . This is simply because that the interfacial stresses can be further intensified (or diminished) by the adjacent material having a higher (or lower) stiffness. The effects of the geometric configuration on the interfacial stresses are displayed in Figs. 5 and 6. It is clear to see that the magnitudes of the interfacial stresses increase with the ratio a_1/b_1 . Based on the above findings, it allows us to find an optimum design of the reinforcement layer such that the magnitude of both stress concentration and the interfacial stresses could be fairly reduced. Note that all these calculated results have been determined by summing up the first ten terms in Eqs. (42) and (43). A good accuracy for the current problem can be demonstrated by the contribution of the leading terms appearing in Eqs. (15) and (16). The contribution of the stresses for the leading terms of a series solution is 25.01%, 10.04%, 1.25% and 0.11%, respectively (see Table 1). The contribution accounts for the ratio of each term to the summation of the first ten terms of a series solution. The leading ten terms have over 99% contribution, making the series solution rapidly convergent. This demonstrates the accuracy and the efficiency of our proposed method. Note that the convergence rate depends on the combinations of material properties and geometric configurations. In general, the convergence rate becomes more rapid if the differences of the elastic

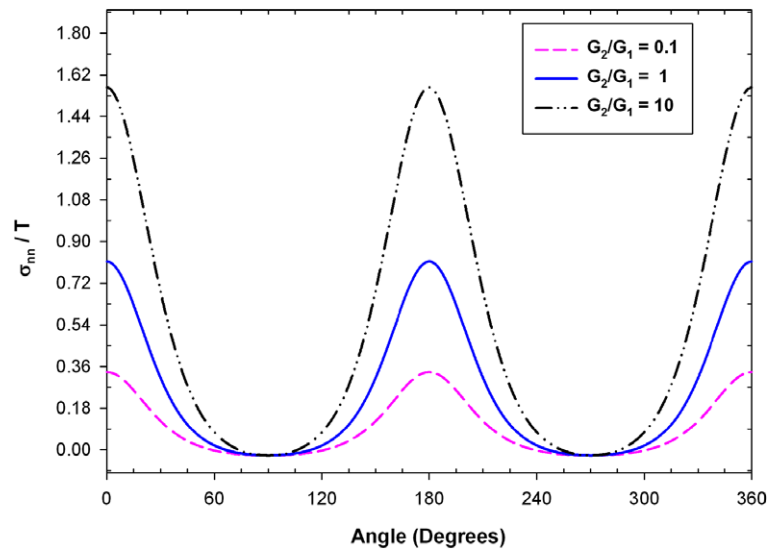


Fig. 3. Angular variations of the interfacial normal stress for different shear modulus ratios. ($a_1/b_1 = 1.5$, $a_2/a_1 = 0.9$, $\nu_1 = \nu_2 = 0.3$, $\gamma = 90^\circ$).

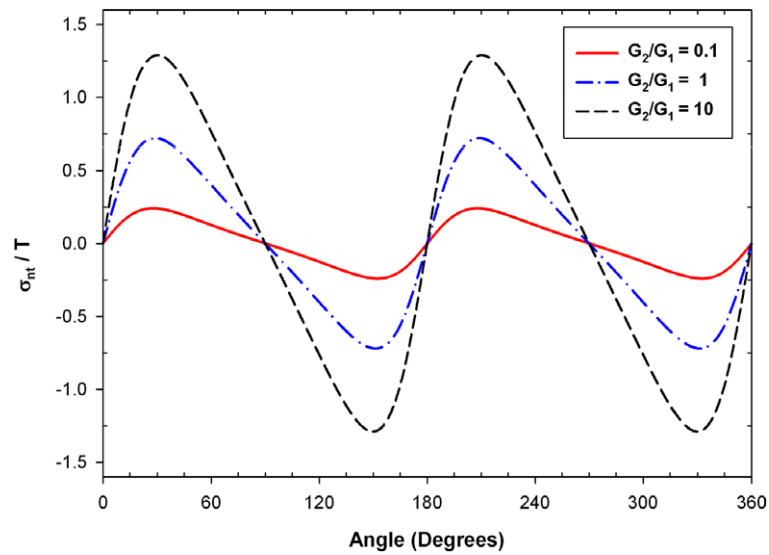


Fig. 4. Angular variations of the interfacial shear stress for different shear modulus ratios. ($a_1/b_1 = 1.5$, $a_2/a_1 = 0.9$, $\nu_1 = \nu_2 = 0.3$, $\gamma = 90^\circ$).

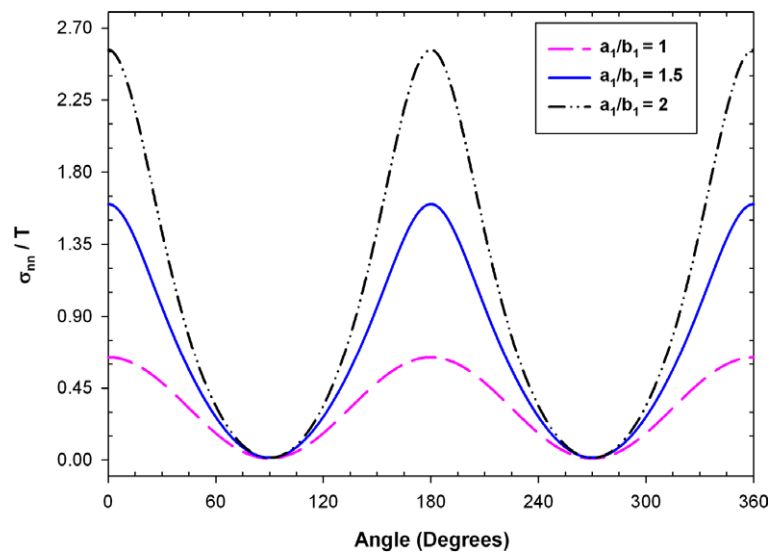


Fig. 5. Angular variations of the interfacial normal stress for different aspect ratios. ($a_2/a_1 = 0.9$, $\nu_1 = \nu_2 = 0.3$, $G_2/G_1 = 10$, $\gamma = 90^\circ$).

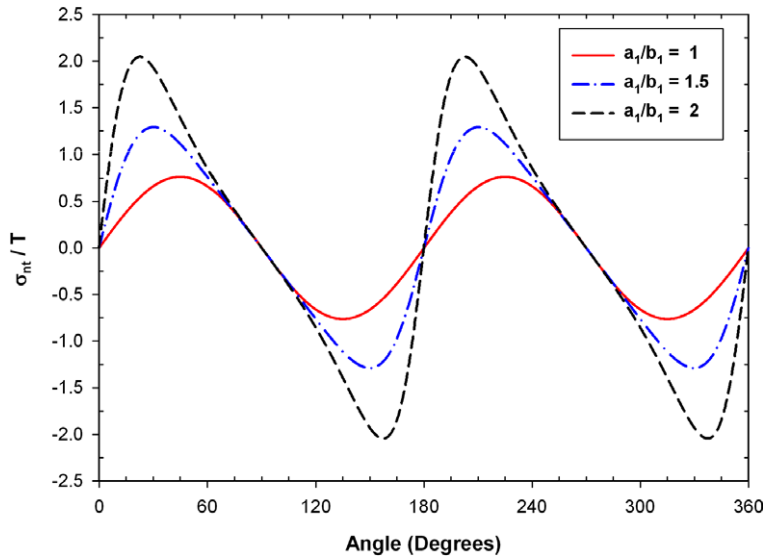


Fig. 6. Angular variations of the interfacial shear stress for different aspect ratios. ($a_2/a_1 = 0.9$, $\nu_1 = \nu_2 = 0.3$, $G_2/G_1 = 10$, $\gamma = 90^\circ$).

Table 1
Contribution of the leading terms $n = 1-10$ for σ_{nn} and σ_{nt} .

Term (n)	Error (%)
1	25.01
3	10.04
5	1.25
10	0.11

constants of the neighboring materials get smaller and the ratio a_1/b_1 (or a_2/b_2) approaches one.

5. Conclusion

The analytical solutions for the problem of a reinforced elliptical hole, embedded in an unbounded matrix and subjected to remote uniform load, are provided in this paper. Compared to the conventional numerical technique, which eventually requires solving a system of simultaneous equations for a large number of unknown constants, the present approach shows more advantages for the analogous problem. Based on the method of conformal mapping and the method of analytical continuation in conjunction with the alternating technique, the elastic fields are obtained as a transformation on the solution to the corresponding homogeneous solution. The present proposed method can also be extended to the problem with any number of layers. Practically, a graded interface can be achieved by multilayered materials with stepwise homogeneous elastic properties. Consequently, the problem with functionally graded materials can be solved using the present proposed method.

Appendix A

When S_1 and S_2 are made of the same material properties, i.e., $\Pi_{12} = \Lambda_{12} = 0$, Eqs. (42) and (43) reduce to

$$(\phi_1(\zeta) = \Pi_{21} \overline{\omega_{0a}} \left(\frac{\rho_1^2}{\zeta} \right) = 0 \quad (A1)$$

$$\phi_{a1}(\zeta) = (1 + \Lambda_{21}) \phi_0(\zeta) + \Pi_{12} \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \zeta}{1 - \frac{\zeta^2}{R^2 \rho_1^4}} c_1 = \phi_0(\zeta) = \frac{T}{4} \zeta \quad (A2)$$

$$\omega_{a1}(\zeta) = (1 + \Pi_{21}) \omega_{0a}(\zeta) + \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \frac{\rho_1^2}{\zeta}}{1 - \frac{\zeta^2}{R^2 \rho_1^4}} \overline{c_1} = \omega_0(\zeta) \quad (A3)$$

$$\omega_1(\zeta) = \Lambda_{21} \overline{\phi_0} \left(\frac{\rho_1^2}{\zeta} \right) - \omega_{0b}(\zeta) + (1 + \Pi_{12}) \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \frac{\rho_1^2}{\zeta}}{1 - \frac{\zeta^2}{R^2 \rho_1^4}} \overline{c_1} = 0 \quad (A4)$$

$$\begin{aligned} \omega_{a1}^I(\zeta) &= \frac{\zeta}{R^2 \rho_1^2} \frac{T}{4} - \frac{T}{2} e^{-2i\gamma \zeta} + \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \frac{\rho_1^2}{\zeta} T}{1 - \frac{\zeta^2}{R^2 \rho_1^4}} \\ &\quad + \frac{\frac{R}{\zeta} (\rho_2^2 - \rho_1^2) + \frac{\zeta}{R} \left(\frac{1}{\rho_2^2} - \frac{1}{\rho_1^2} \right) T}{R - \frac{1}{R \zeta^2}} \frac{T}{4} \\ &= \frac{R \frac{\rho_2^2}{\zeta} + \frac{\zeta}{R \rho_2^2} T}{R - \frac{1}{R \zeta^2}} \frac{T}{4} - \frac{T}{2} e^{-2i\gamma \zeta} \end{aligned} \quad (A5)$$

$$\begin{aligned} \phi_{b1}(\zeta) &= -\overline{\omega_{a1}^I} \left(\frac{\rho_2^2}{\zeta} \right) + \frac{\left(\frac{1}{R^4 \rho_2^4} + 1 \right) \zeta}{1 - \frac{\zeta^2}{R^2 \rho_2^4}} c_1 \\ &= -\frac{R \zeta + \frac{1}{R \zeta} T}{R - \frac{\zeta^2}{R \rho_2^4}} \frac{T}{4} + \frac{T}{2} e^{2i\gamma \frac{\rho_2^2}{\zeta}} + \frac{\left(\frac{1}{R^4 \rho_2^4} + 1 \right) \zeta T}{1 - \frac{\zeta^2}{R^2 \rho_2^4}} \frac{T}{4} \\ &= -\frac{1}{R^2 \zeta} \frac{T}{4} + \frac{T}{2} e^{2i\gamma \frac{\rho_2^2}{\zeta}} \end{aligned} \quad (A6)$$

$$\omega_{b1}(\zeta) = -\overline{\phi_{a1}} \left(\frac{\rho_2^2}{\zeta} \right) - \frac{\left(\frac{1}{R^4 \rho_2^4} + 1 \right) \frac{\rho_2^2}{\zeta}}{1 - \frac{\rho_2^4}{R^2 \zeta^2}} \overline{c_1} = -\frac{T}{4} \frac{\rho_2^2}{\zeta} - \frac{\left(\frac{1}{R^4 \rho_2^4} + 1 \right) \frac{\rho_2^2}{\zeta} T}{1 - \frac{\rho_2^4}{R^2 \zeta^2}} \frac{T}{4} \quad (A7)$$

$$\phi_2(\zeta) = (1 + \Lambda_{12}) \phi_{b1}(\zeta) = \phi_{b1}(\zeta) \quad (A8)$$

$$\omega_2(\zeta) = (1 + \Pi_{12}) \omega_{b1}(\zeta) + (1 + \Pi_{12}) c_2 \frac{\rho_2^2}{\zeta} = \omega_{b1}^I(\zeta) \quad (A9)$$

$$\phi_{a2}(\zeta) = \Pi_{12} \overline{\omega_{b1}} \left(\frac{\rho_1^2}{\zeta} \right) + \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \zeta}{1 - \frac{\zeta^2}{R^2 \rho_1^4}} \Pi_{12} c_2 = 0 \quad (A10)$$

$$\omega_{a2}(\zeta) = \Lambda_{12} \overline{\phi_{b1}} \left(\frac{\rho_1^2}{\zeta} \right) + \frac{\left(\frac{1}{R^4 \rho_1^4} + 1 \right) \frac{\rho_1^2}{\zeta}}{1 - \frac{1}{R^2 \zeta^2}} \overline{c_2} = 0 \quad (\text{A11})$$

In view of Eq. (15) and Eqs. (A1)–(A15), we have

$$\phi(\zeta) = \begin{cases} \phi_0(\zeta) + \phi_1(\zeta) \\ \phi_{a1}(\zeta) + \phi_{b1}(\zeta) = \phi_0(\zeta) + \phi_1(\zeta) \end{cases} \quad (\text{A12})$$

where

$$\phi_0(\zeta) + \phi_1(\zeta) = \frac{T}{4} \zeta - \frac{1}{R^2 \zeta} \frac{T}{4} + \frac{T}{2} e^{2i\gamma} \frac{\rho_2^2}{\zeta} = \frac{T}{4} \zeta + \frac{T}{4} \left[\frac{2e^{2i\gamma}}{\zeta} - \frac{m}{\zeta} \right]$$

where $\rho_2 = \frac{a_2+b_2}{a_2-b_2} = 1$, $R^2 = \frac{a_2+b_2}{a_2-b_2}$, $m = \frac{a_2-b_2}{a_2+b_2}$.

Similarly,

$$\psi(\zeta) = \begin{cases} \psi_0(\zeta) + \psi_1(\zeta) \\ \psi_{a1}(\zeta) + \psi_{b1}(\zeta) = \psi_0(\zeta) + \psi_1(\zeta) \end{cases} \quad (\text{A13})$$

Note that the functions $\omega_0(\zeta)$, $\omega_{a1}(\zeta)$ and $\omega_{b1}^I(\zeta)$ correspond to ρ_1 , but $\omega_{b1}(\zeta)$ corresponds to ρ_2 . Now since we know that

$$\psi_1 = \psi_{b1}(\zeta) = \omega_{b1}(\zeta) - \frac{R \frac{\rho_2^2}{\zeta} + \frac{\zeta}{R \rho_2^2}}{R - \frac{1}{R \zeta^2}} \phi_{b1}'(\zeta) \quad (\text{A14})$$

Then with the aid of (A14) and (A13) reduces to

$$\begin{aligned} \psi(\zeta) &= -\frac{T}{2} e^{-2i\gamma} \zeta - \frac{T}{4} \frac{1}{\zeta} - \frac{\left(\frac{1}{R^4} + 1 \right) \frac{1}{\zeta} T}{1 - \frac{1}{R^2 \zeta^2}} \frac{T}{4} \\ &\quad - \frac{R^2 \zeta + \zeta^3}{R^2 \zeta^2 - 1} \left(\frac{1}{R^2 \zeta^2} \frac{T}{4} - \frac{T}{2} e^{2i\gamma} \frac{1}{\zeta^2} \right) \\ &= -\frac{T}{2} e^{-2i\gamma} \zeta - \frac{T}{4} \frac{1}{\zeta} - \frac{(m^2 + 1) \zeta T}{\zeta^2 - m} \frac{T}{4} \\ &\quad - \frac{\zeta + m \zeta^3}{\zeta^2 - m} \left(\frac{m T}{\zeta^2} \frac{T}{4} - \frac{T}{2} e^{2i\gamma} \frac{1}{\zeta^2} \right) \end{aligned} \quad (\text{A15})$$

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